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A Mathematical Model for the Effect of Earthquake on High Rise Buildings of Different Shapes

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ABSTRACT

The paper explores the effects of mechanical vibrations of multi-storey buildings. The vibrations are induced by earthquake. The aim of the paper is to apply the theoretical knowledge of differential equations to study the effect of earthquake on multi-storeyed buildings. The main aim of this paper is to study and deduce the stability of buildings with different dimensions and shapes during an earthquake. The study involves a system of differential equations that are solved by applying the method of eigenvalues and eigenvectors. The effect of Coulomb damping is also incorporated in the equations.

Keywords: System of differential equations; Mathematical modeling; Earthquake; Multi-storeyed building; Coulomb damping

INTRODUCTION

As seismic waves move through the ground, the ground also moves at its natural frequency. During an earthquake if the frequency with which the building sways, matches with building's natural frequency, i.e. when the frequency contents of the ground motion are centered around the building's natural frequency, we say that the building and the ground motion are in resonance with one another. Resonance tends to increase or amplify the building's response. Because of this, buildings suffer the greatest damage from ground motion at a frequency close or equal to their own natural frequency.Although the phenomenon of resonance can be extremely damaging, its effects can be reduced. In designing seismically safe buildings, an architect or engineer must be concerned with "tuning" a building so that the tendency for its own vibration to be amplified by resonance is reduced or eliminated. There come hundreds of small earthquakes around the world every day. Some of them are so minor that we, humans cannot even feel them, but seismographs and other sensitive machines can record them. However, some of them result in great devastation, taking the life of thousands of people, and destroying the properties of billions. Today, it has become imperative that structures should be designed to resist earthquake forces, in order to reduce the loss of life. The science of Earthquake Engineering and Structural Design has improved tremendously. Architects around the world are trying their best to design safe structures that can withstand earthquakes of reasonable magnitude.

A common misconception is that a taller building will face more damage than a shorter building. The Mexico City earthquake of September 19, 1985 provides a striking illustration to contradict this. A majority of the buildings that collapsed during this earthquake were around twenty storey tall. These twenty storey buildings were in resonance with the frequency contents of the 1985 earthquake. Other buildings, of different heights (some greater than twenty storeys) and with different vibration characteristics, were often found undamaged even though they were located right next to the damaged twenty storey buildings.

The mathematical conceptualization of the vibrations of an earthquakehelp to understand various questions that may arise. Which building will collapse first compared to the other building? What period of an earthquakewill produce the destructive resonance vibration onthe building? Why an earthquake will demolish one building but leave the one nextdoor untouched? Although it is difficult to incorporate all the causes that bring about the damages in the building due to earthquake, qualitative answers to these questions can be got by modeling the problem by homogeneous system of second order differential equations that are also applied to model various mechanical applications. The system of spring coupled masses [1, 2] can be used to investigate the response to transverse earthquake ground oscillations of multi-storeyed buildings. In this paper, the eigenvalues method for homogeneous systems is applied to investigate the movement of various floors of the multi-storeyed buildings by incorporating the effect of earthquake as periodic forced vibrations. The eigenvalues are used to find the natural frequency (Ω) of each floor and its respective time period. By examining the values ofΩ, it can be estimated that what magnitude and period of an earthquake will affect a particular floor or a building leaving the others untouched. Corresponding to the mathematical model for mass- spring system, which represents the simple harmonic motion; same theory is applied to formulate the mathematical model for the effect of earthquake induced vibrations on the buildings by making an assumption about the structure of the building and restricting the motion of the floors in the horizontal direction only. In this paper, some of the basic structures of building are considered and analyzed for the effect of earthquake.

METHODOLOGY

In order to model the structural dynamics of a building consisting of *n* floors, we make the following assumptions:

(i) The floors have masses m_1, m_2, m_3, …,m_n. Each floor is assumed to be a point mass concentrated in the centre of each floor.

(ii) A linear restoring force acts on each floor that is incorporated in the model by the stiffness factor k_1, k_2, k_3, …,k_n.

(iii) There is a damping force which is directly proportional to the damping constants c_1, c_2, c_3, …,c_n between the floors.

(iv) A horizontal earthquake oscillation, $Ecos\omega t$ of the ground with amplitude E and acceleration $a = -E\omega^2 cos\omega t$, produces a force $F = ma = m\,E\omega^2 cos\omega t$ on each floor of the building.

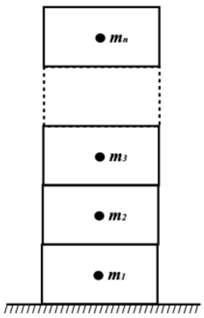


Figure I: Mathematical model of the building

Applying Newton's second law to the floors of the building yield the equation of motion as:

$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) - c_1(x_1' - x_2') + m_1E\omega^2 cos\omega t$$
$$m_2x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2) - c_1(x_2' - x_1') - c_2(x_2' - x_3') + m_2E\omega^2 cos\omega t$$
$$m_3x_3'' = -k_3(x_3 - x_2) + k_4(x_4 - x_3) - c_2(x_3' - x_2') - c_3(x_3' - x_4') + m_3E\omega^2 cos\omega t$$
$$m_nx_n'' = -k_n(x_n - x_{n-1}) - c_{n-1}(x_n' - x_{n-1}') + m_nE\omega^2 cos\omega t \quad (1)$$

where $x_1, x_2, \dots$ are displacement from the equilibrium state.

In terms of displacement vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$,

he mass matrix $M = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix}$,

the stiffness matrix $K = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 & \cdots & 0 & 0 \\ k_2 & -(k_2+k_3) & k_3 & \cdots & 0 & 0 \\ 0 & k_3 & -(k_3+k_4) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -(k_{n-1}+k_n) & k_n \\ 0 & 0 & 0 & \cdots & k_n & -k_n \end{bmatrix}$

and the damping matrix $C = \begin{bmatrix} -c_1 & c_1 & 0 & \cdots & 0 & 0 \\ c_1 & -(c_1+c_2) & c_2 & \cdots & 0 & 0 \\ 0 & c_2 & -(c_2+c_3) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -(c_{n-1}+c_n) & c_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & -c_{n-1} \end{bmatrix}$

the system in equation (1) can be written in the matrix form as

$$M\overrightarrow{x''} = C\overrightarrow{x'} + K\vec{x} + \vec{F} \quad (2)$$

In equation (2), the forced vibration vector $\vec{F} = \begin{bmatrix} m_1 E\omega^2 \cos\omega t \\ m_2 E\omega^2 \cos\omega t \\ m_3 E\omega^2 \cos\omega t \\ \vdots \\ m_n E\omega^2 \cos\omega t \end{bmatrix}$

The diagonal matrix $\boldsymbol{M}$is non- singular and to get its inverse, $\boldsymbol{M^{-1}}$replaces each diagonal element with its reciprocal. Multiplying each side of equation (2) by $\boldsymbol{M^{-1}}$ gives the second-order system

$$\vec{x}'' = P\vec{x}' + A\vec{x} + \vec{f} \tag{3}$$

where $A = M^{-1}K$, $P = M^{-1}C$ and $\vec{f} = M^{-1}\vec{F}$.

We first find the solution of the homogeneous equation

$$\vec{x}'' - P\vec{x}' - A\vec{x} = 0 \tag{4}$$

In order to find a solution of the equation (4), substitute a trial solution of the form

$$\vec{x}(t) = \vec{v}e^{\alpha t} \tag{5}$$

where $\vec{v}$ is a constant vector. Substituting equation (5) in equation (4) gives

$$\left(\alpha^2 I - \alpha P - A\right)\vec{v}e^{\alpha t} = 0 \tag{6}$$

where$\boldsymbol{I}$ is the identity matrix of dimensions same as $P$and A.

For a non-trivial solution we require

$$\det\left(\alpha^2 I - \alpha P - A\right) = 0 \tag{7}$$

Alternatively, the system in equation (4) can also be solved by reducing the second order system to a system of two first order system of equations.

For this, substitute $\boldsymbol{x}' = \boldsymbol{y}$in equation (4) and system of two first order system of equation can be written as:

$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & P \end{bmatrix}\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} \Rightarrow X = DX \tag{8}$$

where$X$ and D are 2 x 1 and 2 x 2 block matrices whose entries are nx 1and nxn matrices respectively.

The system can be solved using the eigenvalues method for homogeneous systems by finding the eigenvalues for the block matrix Dgiven by det $(D - \lambda I) = 0$.

$$\det\begin{bmatrix} -\lambda\,\boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{A} & \boldsymbol{P} - \lambda\,\boldsymbol{I} \end{bmatrix} = 0 \tag{9}$$

We state here a result proved in [3],

If $M=\begin{bmatrix} S & T \\ U & V \end{bmatrix}$, *where S, T, U, V are block matrices such that any of the blocks S and T or S and U or U and V or T and V commute then* det (M)= det $(SV - UT)$.

From the matrix in (9) it is trivial to see that blocks S and U; S and T; T and V commute. Therefore the determinant of (9) is given by

$$\det\left(\lambda^2 \boldsymbol{I} - \boldsymbol{P}\lambda - \boldsymbol{A}\right) = 0 \tag{7'}$$

which is the same result as in equation (7).

In the absence of the Coulomb damping (C = 0), the eigenvalues $\lambda_i^2$of the system given by equation (7’) are negative.
Let λ_1^2, λ_2^2, λ_3^2, …,λ_n^2 be the eigenvalues of the matrix $A = M^{-1}K$. Then we have

$$\sum_{i=1}^{n} \lambda_i^2 = \text{trace(A)} \text{and} \prod_{i=1}^{n} \lambda_i^2 = \det(\text{A})$$

$$\sum_{i=1}^{n} \lambda_i^2 = -\left(\frac{k_1}{m_1} + k_2\left\{\frac{1}{m_1}+\frac{1}{m_2}\right\} + k_3\left\{\frac{1}{m_2}+\frac{1}{m_3}\right\} + \ldots + k_n\left\{\frac{1}{m_{n-1}}+\frac{1}{m_n}\right\}\right) < 0$$

$$\prod_{i=1}^{n} \lambda_i^2 = \det(M^{-1}K) = \det(M^{-1})\det(K) = \frac{(-1)^n}{\prod_{i=1}^{n} m_i}\prod_{i=1}^{n} k_i$$

It can be easily proved using mathematical induction that each λ_i^2 is negative. Also
M^{-1} is a diagonal matrix and K is a symmetric tridiagonal matrix. Therefore each λ_i^2 is real and distinct.
Unlike the case of absence of Coulomb damping [4], where there were only real negative values of the eigenvalues, in the presence of Coulomb damping the eigenvalues λ_I may either be real or complex.The nature of the eigenvalues will decide the type of damping occurring in the system- over damped, critically damped or under damped. *In the under damped case, when the eigenvalues comes out to be complex conjugate roots, the real part of the eigenvalues forms a part of amplitude of the oscillation exponentially decreasing with time and the imaginary part gives the circular frequencyΩ (pseudo frequency) of the oscillation.*Therefore, with the Coulomb damping, floors may start and stop vibrating several times before the entire building comes to rest.
The system (3) can then be written as:

$$\overrightarrow{x''} - P\overrightarrow{x'} - A\vec{x} = \left(E\omega^2 \cos\omega t\right)B \qquad (10)$$

where $B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}^T$

In order to find a particular solution, we use the method of undetermined coefficients,
$x_p(t) = \boldsymbol{c}\cos\omega t + \boldsymbol{d}\sin\omega t$ (11)
with the known external frequency ω of the earthquake. In equation (11) $\boldsymbol{c}$ and $\boldsymbol{d}$ are constant column vectors.
Substituting (11) in (10) and equating the coefficients we get
$\boldsymbol{d} = \omega(A + I\omega^2)^{-1}P\,\boldsymbol{c}$ (12) $\boldsymbol{c} = [\omega^2 P(A + I\omega^2)^{-1}P + (A + I\omega^2)]^{-1}F_0$ (13)
where$\boldsymbol{F_0} = -E\omega^2 B$
In the case of damping there are no phenomena of pure resonance where the natural frequency matches the frequency of the earthquake. However, what we talk about here is called the practical resonance where forced amplitude of vibration of the building remains finite for the value of ω (frequency of earthquake) but might attain a maximum. The frequency of earthquake does not exactly match the natural frequency of the building but is very close to it.

RESULTS

Although most of the high rise building are rectangular in shape, we can classify them in the following categories

(i) Mass of each floor is constant (rectangular shape)

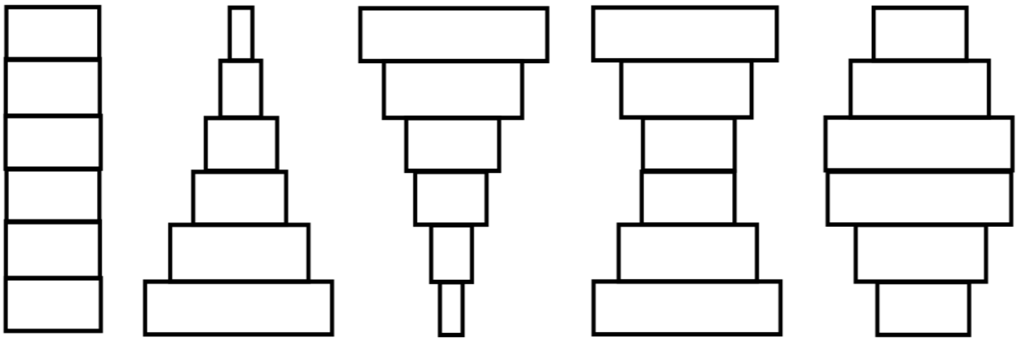
(ii) Mass of the floor decreases as we go up (pyramidal shape) Eg. pyramid-shaped tower to be built in the center of Jerusalem, Tour triangle proposed to be built in Paris

(iii) Mass of the floor increases as we go up (inverted pyramidal shape) Eg. Hanoi Museum, Slovak radio building

(iv) Mass of the floor first decreases and then increases as we go up (cross shape) Eg. Disaster education centre at Jerusalem

(v) Mass of the floor first increases and then decreases as we go up (hexagonal shape) Eg. The national library, Minsk Brussel, Aldar headquarters, Abu Dhabi

Geometrically these buildings may be represented as



(i) (ii) (iii) (iv) (v)

Mathematically the entries in the mass matrix M will be

(i) $m_1 = m_2 = m_3 = \ldots = m_n$

(ii) $m_1 > m_2 > m_3 > \ldots > m_n$

(iii) $m_1 < m_2 < m_3 < \ldots < m_n$

(iv) $m_1 > m_2 > m_3 > \ldots > m_p$; $m_p < m_{p+1} < m_{p+2} < \ldots < m_n$

(v) $m_1 < m_2 < m_3 < \ldots < m_p$; $m_p > m_{p+1} > m_{p+2} > \ldots > m_n$

For a rectangular building, the mass of each floor is taken as 1000 units. For the other cases the mass is increased or decreased by 200 units. The stiffness parameters k_i's are taken as constant for each floor with a value of 10000 units and the damping parameters c_i's are also taken as constant for each floor with a value of 500 units. The oscillation frequencies and time-period for buildings of seven floors and four floors for different cases are calculated using equation (7) and tabulated in Table I – Table V

Table-I Angular frequency (Ω) and time-period (T) for a rectangular building

With damping				Without damping			
Seven Floors		Four Floors		Seven Floors		Four Floors	
Ω	T	Ω	T	Ω	T	Ω	T
6.11	1.03	5.87	1.07	0.66	9.50	1.10	5.72
5.71	1.10	4.81	1.31	1.95	3.21	3.16	1.99
5.07	1.24	3.15	1.99	3.16	1.99	4.84	1.30
4.20	1.49	1.10	5.72	4.23	1.48	5.94	1.06
3.15	1.99	-	-	5.12	1.23	-	-
1.95	3.22	-	-	5.78	1.09	-	-
0.66	9.51	-	-	6.19	1.02	-	-

Table-II Angular frequency (Ω) and time-period (T) for a pyramidal shape building

With damping				Without damping			
Seven Floors		Four Floors		Seven Floors		Four Floors	
Ω	T	Ω	T	Ω	T	Ω	T
5.41	1.16	5.40	1.16	0.49	12.96	0.92	6.85
4.64	1.35	4.30	1.46	1.56	4.02	2.82	2.23
4.07	1.54	2.81	2.24	2.55	2.47	4.33	1.45
3.39	1.85	0.92	6.85	3.40	1.85	5.45	1.15
2.54	2.47	-	-	4.09	1.54	-	-
1.56	4.02	-	-	4.67	1.35	-	-
0.48	12.96	-	-	5.46	1.15	-	-

Table-III Angular frequency (Ω) and time-period (T) for an inverted pyramidal shape building.

With damping				Without damping			
Seven Floors		Four Floors		Seven Floors		Four Floors	
Ω	T	Ω	T	Ω	T	Ω	T
5.20	1.21	5.19	1.21	0.57	11.07	1.01	6.20
4.49	1.40	4.16	1.51	1.56	4.04	2.75	2.28
3.95	1.59	2.75	2.29	2.49	2.52	4.18	1.50
3.30	1.91	1.01	6.20	3.31	1.90	5.23	1.20
2.48	2.53	-	-	3.97	1.58	-	-
1.55	4.04	-	-	4.52	1.39	-	-
0.57	11.07	-	-	5.25	1.20	-	-

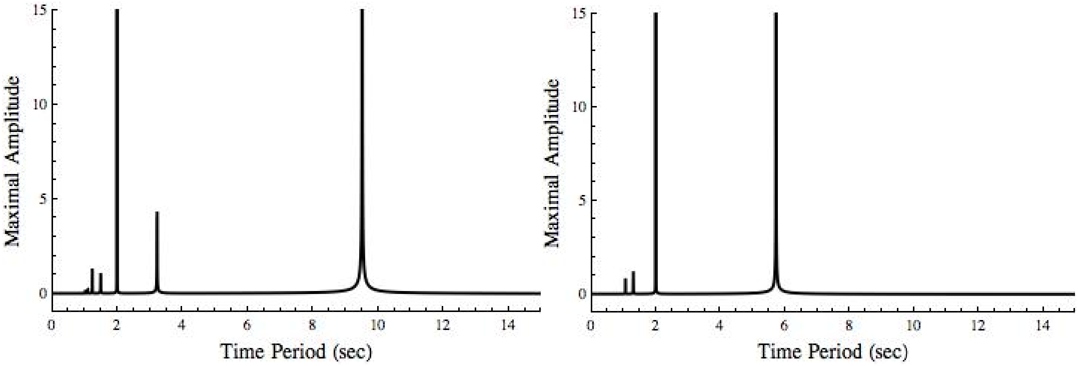
Table-IV Angular frequency (Ω) and time-period (T) for a cross shape building.

With damping				Without damping			
Seven Floors		Four Floors		Seven Floors		Four Floors	
Ω	T	Ω	T	Ω	T	Ω	T
5.69	1.10	5.77	1.09	0.57	11.00	1.05	5.99
4.93	1.27	4.56	1.38	1.67	3.77	2.97	2.12
4.37	1.44	2.96	2.12	2.72	2.31	4.59	1.37
3.59	1.75	1.05	5.99	3.61	1.74	5.84	1.08
2.72	2.31	-	-	4.40	1.43	-	-
1.67	3.77	-	-	4.97	1.26	-	-
0.57	11.00	-	-	5.75	1.09	-	-

Table-V Angular frequency (Ω) and time-period (T) for a hexagonal shape building.

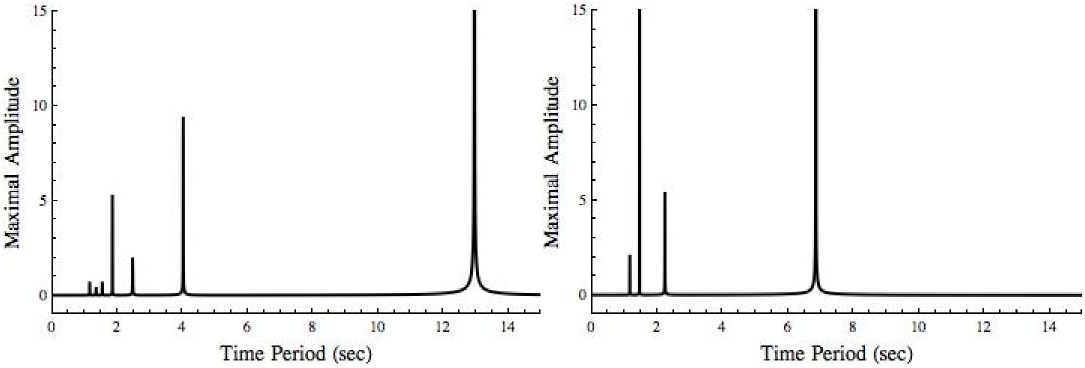
With damping				Without damping			
Seven Floors		Four Floors		Seven Floors		Four Floors	
Ω	T	Ω	T	Ω	T	Ω	T
5.42	1.16	5.51	1.14	0.59	10.69	1.05	6.01
5.19	1.21	4.65	1.35	1.78	3.54	3.06	2.05
4.53	1.39	3.05	2.06	2.85	2.20	4.68	1.34
3.81	1.65	1.04	6.01	3.83	1.64	5.57	1.13
2.84	2.21	-	-	4.56	1.38	-	-
1.77	3.54	-	-	5.23	1.20	-	-
0.59	10.69	-	-	5.47	1.15	-	-

From Table I one concludes that though an earthquake of time-period approximately 2 sec will be resonant with both seven storeyed and four storeyed building. But an earthquake with time-period approximately 5.7 sec will resonate with a four storeyed building (with damping case). This phenomenon for the case of without damping is also observed in the maximal amplitude versus time-period of oscillation of the earthquake that is plotted in Figure II – Figure VI (using equation 7 with P = 0). Also an earthquake with time-period approximately 5.7 sec will be more destructive for a four storeyed building as compared to an earthquake of time-period 3.54 sec that resonates with a seven storeyed building.



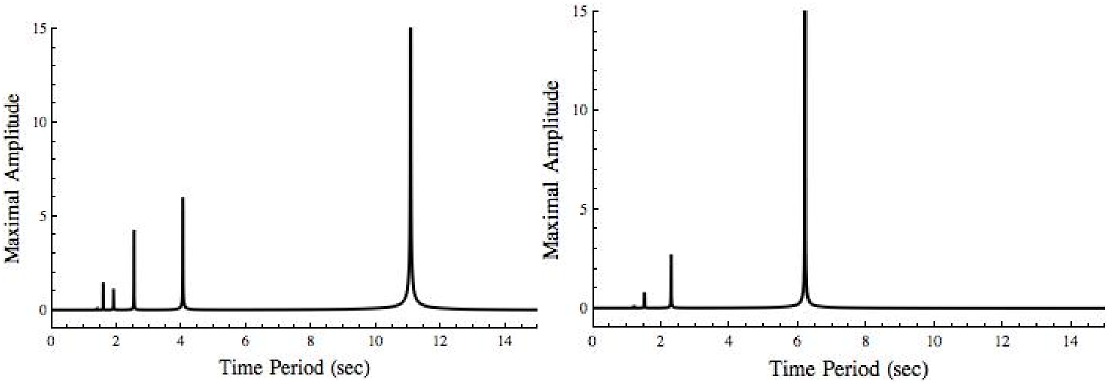
(a) (b)

Figure-II: Maximal amplitude versus time-period of oscillation for a rectangular building (a) Seven Storeyed (b) Four Storeyed



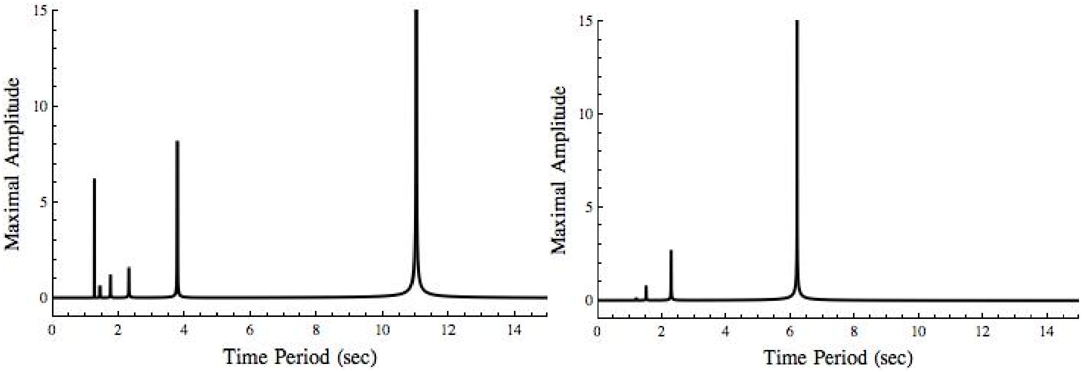
(a) (b)

Figure-III: Maximal amplitude versus time-period of oscillation for a pyramidal shape building (a) Seven Storeyed (b) Four Storeyed



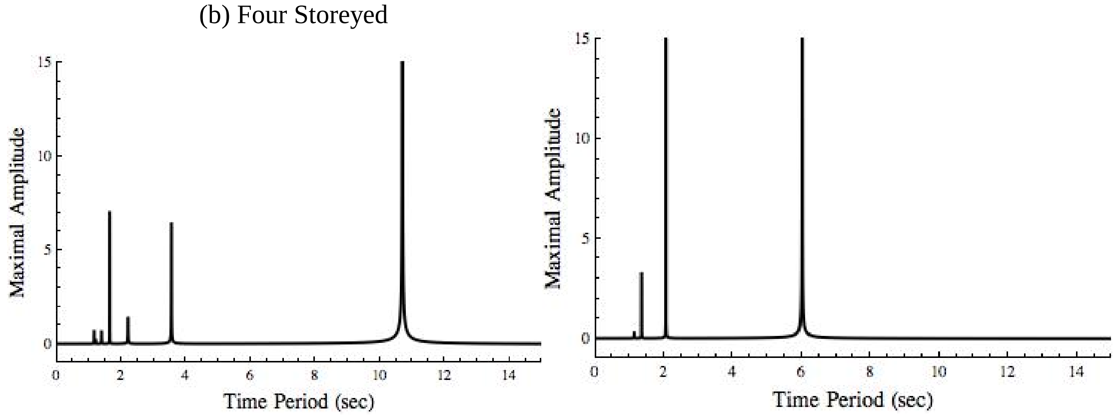
(a) (b)

Figure-IV: Maximal amplitude versus time-period of oscillation for aninverse pyramidal shape building (a) Seven Storeyed (b) Four Storeyed



(a) (b)

Figure-V: Maximal amplitude versus time-period of oscillation for a cross shape building (a) Seven Storeyed (b) Four Storeyed



(a) (b)

Figure-VI: Maximal amplitude versus time-period of oscillation for a hexagonal shape building (a) Seven Storeyed (b) Four Storeyed

CONCLUSION

A mathematical model based on forced spring oscillation is extended to study the effect of earthquake on high-rise buildings of different shapes. In order to get a qualitative understanding of the problem the mass of each floor is assumed to be located at the mass centre of each floor. Further we assume that the stiffness and damping parameters for each floor are constants. The model gives a clear insight as to why an earthquake of certain frequency effect adversely on a building even though it may have a lesser number of floors.

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